George Peacock: Treatise on Algebra, 1830, 730pp. The original Preface is long and verbose, 30 pages, v-xxxv.
Condensed Preface is 2,055 words, (2 pages, at c.1000 words/page), by Assad Ebrahim 2020-04-01---Wed to 2020-04-06---Mon

URL: www.mathscitech.org/articles/
email: assadebrahim2000@gmail.com

=== Preface to "A Treatise on Algebra" by George Peacock, 1830

[1] ALGEBRA HAS MISTAKENLY BEEN CONSIDERED AS AN UNRESTRICTED SYMBOLICAL ABSTRACTION of Arithmetic in which the operations of arithmetic are transferred to Algebra without any need to re-state either their meaning or application. The symbols of Algebra are assumed to be the general and unlimited representation of all quantity, and the operations of Addition and Subtraction (denoted by signs + and -) able to be freely used in connecting such symbols with each other in any symbolic expression, without any restrictions or modification to their meaning or scope, and similarly for Multiplication and Division. (p.vi) But this is not the case, and there are problems that arise with the unrestricted usage of symbols to represent unspecified quantities with arithmetic operations that have precise and restricted meaning. (p.vii)
Firstly, + and - need be confined to quantities of the same kind. [For example: it is not correct to add 2 eggs + 1 dozen eggs, or to add lines and areas, or times and velocities.] Secondly, subtraction is valid only when the first number is greater than the second. [Example: 3 - 5 is not valid, as one cannot take away 5 from 3 without making explicit some other concept, such as a debt of 2.] (p.vii)
A third challenge is that of unrestricted substitution. Consider:
c := (a+b), and a-c.
Both expressions look valid. But substituting gives:
a-(a+b) = a-a + a-b = a-b
which is only valid if a>b. This restriction is not apparent in the 2nd expression. (p.ix)

[2] WHAT WE WILL DO IN THIS TREATISE IS TO FREE THE SYMBOLS + AND - FROM THEIR RESTRICTIONS IN ARITHMETIC and establish them as new symbols with no meaning until we have indicated this, i.e. they must have meaning that is independent of any laws of arithmetic in order to avoid the inconsistencies which have led to the critique of Algebra so far. (p.ix) In this way, this work aims to show that Algebra can be a demonstrative science, and to resolve these recent critiques of the logical imperfections in Algebra as it is currently explained. (p.v-vi)

[3] THE KEY OBSERVATION IS THAT THE CONNECTION OF ALGEBRA TO ARITHMETIC IS *BY CONVENTION*, and not mathematically nor logically necessary, i.e. Algebra has been inspired by suggestion by Arithmetic, not founded upon it. The operations and laws of symbolical combination are assumed, not arbitrarily, but with a general reference to their anticipated interpretation within a subordinate science. An important consequence is that the laws for combination of symbols are completely separated from their interpretation. (pp.xx)

[4] WHEN CONSIDERING THE INTERPRETATION OF OPERATIONS, WE MUST ONLY LOOK AT THE FINAL RESULT, and not look to interpret every intermediate part of the transformation process, for while the connection between successive forms is algebraically necessary, the algebraic laws that govern them are independent of their interpretation. In this way Algebra becomes accommodated to the form and peculiar character of any subordinate science: to Arithmetic which is the most common algebraic system, but also to Geometry [as Vector Algebra] after defining through the very comprehensive sign sqrt{-1} the relations of line segments to each other with respect to both magnitude (length) and direction (angle); to Mechanics and Dynamics by defining forces as vectors; and similarly to any other branch of precise investigation, which can be reduced to fixed and invariable principles, or laws of operation. (p.xxi) [AE: Indeed it is possible to define the symbols and operations in some other manner provided so long as these definitions do not lead to contradictions, as the work of Boole and Hamilton and others (in the years to come) will show.]

[5] WITHIN ALGEBRA, MISUNDERSTANDINGS ARISE FROM beginning with the meaning of operations and attempting to make the results obtained dependent upon this meaning. This leads constantly to results which are at variance with such interpretations, for example the situation of numbers with negative signs, but even more problematic is the square- and even- roots of a negative quantity which violates the sign rules for multiplication. (p.xxvi)

[6] BY ALLOWING THE SYMBOLIC EXISTENCE of the sign sqrt{-1} within the essential generality of Algebra as an uninterpreted symbolical language, we can then adduce a meaningful interpretation, as has been done by Adrien Buee (1806) who interpreted sqrt{-1} as indicating perpendicularity in Geometry. This was followed by John Warren (1828) who gave the geometric interpretation of roots of unity in the context of line segments (vectors) in Geometry. (p.xxvii) This line of inquiry sees its culmination in the observation that cos(theta) + i sin(theta) provides the continuous two-dimensional representation of a unit vector through the plane, representable through the work of Euler as e^{i theta), the general complex number. Indeed, we are be able to interpret expressions such as a+b(sqrt{-1}) and a-b(sqrt{-1}), a(cos(\theta) + (sqrt{-1})sin(\theta)) once only the symbolical laws of the sign (sqrt{-1}) have been determined. (p.xxviii-xxxi) This singular geometric sign make it necessary to incorporate the science of Trigonometry within Algebra, taking Geometry as the underlying analogy, just as previously this was done for Arithmetic. There is a very deep connection between Geometry and Algebra which deserves the minute examination of the definitions and first principles of Geometry so that it may be superseded by Algebra. (p.xxxii-xxxiii)

[7] WE THUS CONCLUDE THAT THE SYMBOLICAL ALGEBRA IS A SUBJECT THAT IS MUCH BROADER than Arithmetic, and not limited nor determined by its laws, but can take these on and in doing so provide much service to the mathematician in solving general arithmetic problems [such as the finding of roots and solving of equations]. (p.viii) [AE: Peacock's Treatise on Algebra has broad scope including theory of equations, combinatorics, probability, trigonometry, logarithms, polynomials, and imaginary roots.]

===

[8] THE PRINCIPLE OF THE PERMANENCE OF EQUIVALENT FORMS, which appears to me so important in generalizing the results of algebraical operations, derives its authority from taking the principles of Algebra by

analogy with the those of Arithmetic. The principle assumes the operations of Algebra and their results as independent of the specific values of the symbols, and of the equivalent forms as existing whatever values their symbols may be supposed to possess, yet maintains the result and meaning with arithmetic whenever their their value and interpretations coincide.

[9] To preserve the accordance between the two sciences, we specifically endow the symbolical algebra and its operations with the same rules as the arithmetical algebra so that wherever the two have the same expression they give the same results. This means that we may legitimately assume that the operations in symbolical algebra that are denoted by +, -, x, and /, coincide strictly in meaning with the operations of Addition, Subtraction, Multiplication, and Division in Arithmetic whenever the quantities submitted to them are arithmetical. It is only when the quantities which are subjected to such operations are of a different nature that we are obliged to seek for an interpretation, when possible, of the meaning of such operations and of their results. (p.xii)

[10] To achieve this we frame the definitions of algebraic operations to regard only the LAWS governing their combination of symbols. This will hold for +,-,x,/. To gain the power of further simplification, we then specify the rules that govern these new abstract symbols. Firstly, + and - are inverses, x and / are inverses. By this we mean that:
a+b-b = a   a-b+b = a   ab/b = a   (a/b)b = a   [AE: this requires a law of operation precedence or parentheses] (p.x)
Secondly, we need the sign = to signify the result of or algebraically equivalent to.

[11] Continuing: the double of a quantity can be a+a, -a -a or -a + (-a) since these are quantities of the same kind, whatever they may denote. Also 2a = a+a, and 2(-a)=-2a = -a -a. In a similar manner we may derive the rule for the collection of like terms into one: 5a+3a = 8a since a+a+a+a+a+(a+a+a) = a+a+a+a+a+a+a+a = 8a. Similarly 5a-3a=2a. Reductions like these may be easily generalized so as to lead to the common rule which is given for effecting them. (p.xiii/xiv)

It must also be considered that symbols can be of same or different kinds, e.g. ab or -ab when a and be are lines, or when one of them is a line and another is an area, or one is a time and the other a velocity.

[12] Thus the expression -b + a = (algebraically) a + (-b) = a-b. Now if a and b are quantities of the same kind and a>b then a-b has an immediate and simple interpretation. Similarly -5a=5(-a) which means 5 times (-a) so that must also be the interpretation of -5a for consistency. (p.xiv/xv)

[13] But just as there are multiple ways in arithmetic to write 24, i.e. 12x2 or 8x3 or 6x4 or 2x3x4, so in algebra we have a-b which can be written as a + (-b) or a - (+b). And ab may arise from +a(+b) or (-a)(-b). -ab may be from +a(-b) or (-a)b and similarly in other instances. Just as there is no preference in arithmetic for which of the equivalent forms is used, there is no preference is algebra. The interpretations are to be determined from the result alone and not from the primitive elements. (p.xv)

[14] With these definitions we can reduce the use of ordinary language. [AE: Note in this form we have essentially identified key aspects of group/ring theory, namely an object set (quantities) allowing operations of combination that are closed and that each have inverses.] (p.xi)

[15] Repetition of operation of addition has been shown to be by multiplying the symbol affected by the number or coefficient which is equal to the number of repetitions of the symbol itself. In multiplication, repetition is denoted by writing the symbol with an index (superscript, also called power) equal to the number of repetitions. These simplifications of expressions are in both cases independent of the specific nature of the quantity which the symbol denotes. So we can take as governing law the same that holds for arithmetic, i.e.
ma +na = (m+n)a for addition and a^m * a^n = a^{m+n}.

[16] We see this principle in use in establishing for example the index rule: (a^m)^n = a^(mn) and also in the Binomial Theorem, where the index is a genral symbol. (p.xvii/xviii)

[17] We invoke the PRINCIPLE OF THE PERMANENCE OF EQUIVALENT FORMS so that these same laws hold when m and n are general symbols i.e. of any signs whatsoever, whether fractional or negative.

[18] From this we get also the principle of indices (called in modern terms power rules) and it becomes therefore the general principle which determines the interpretation of indices. (p.xvi/xvii)

[19] One particular point is of great importance but also peculiar delicacy and difficulty:

[20] When we write (1+x)^n, the meaning is clear when n is a whole number, i.e. this is the repeated multiplication of (1+x). As we have seen, this too, may be problematic for arithmetic, but the more important issue is how to interpret the expression when n is a general symbol. In this case, we refer to the power law, even if we cannot say that this is repeated multiplication, and we are indifferent whether or not the symbol has a meaning that accords it arithmetic existence. It suffices that the laws of algebra, which match those of arithmetic by their design, may be executed, and we withhold interpretation till the symbolic calculations are complete. (p.xviii)

[21] This is of paramount importance since the discovery of important properties and relations constitutes a great part of the substance of algebraic investigations and of the artifices of Algebra. All such forms need not exist by arithmetical necessity, and indeed any specific arithmetic connection does not influence the algebraic conclusions.

[22] This holds when we consider functions u(x) and their symbolic derivatives Du. (p.xix). It holds also for the theory of the binomial and polynomial theorems and the theory of series, all of which are in this respect difficult to reconcile logically. [AE: indeed, this was also Fourier's challenge, and which would trigger the investigations of Cantor and Weierstrass and Dedekind in the 2nd half of the 1800's]. (p.xxv)
The other subject which is treated very differently in symbolical vs. arithmetical algebra is the theory of root finding of polynomial equations.